Compactness for Classical Propositional Logic

Definition 1 (Compactness).

Let a consequence relation $\vdash$ for some logic L be given. $\vdash$ is *compact* if, and only if:

$$\Gamma \vdash x \implies \exists \Gamma_{fin} \subseteq \Gamma \text{ such that } \Gamma_{fin} \vdash x$$

where Γ_{fin} is a finite subset of Γ.

In this note, we are interested in compactness for classical propositional logic. We will treat $\vdash$ semantically.

Definition 2 (Classical Propositional Logic and Semantic Consequence ($\vdash$)).

Let $\mathcal{L}$ be our standard propositional language, $\mathbb{V}$ be the set of all possible valuations (worlds) $v : \Phi \to \{0, 1\}$, and $\models$ the usual satisfaction relation on $\mathbb{V} \times \mathcal{L}$.

We define classical consequence $\vdash$ as follows:

$$\Gamma \vdash \phi \iff \text{For all } v \in \mathbb{V} : \text{if } v \models \gamma \text{ for all } \gamma \in \Gamma, \text{then } v \models \phi$$

(Note: Many logic textbooks use $\models$ for semantic consequence as well as for the satisfaction relation $\models \subseteq \mathbb{V} \times \mathcal{L}$.)

Because of how this consequence relation is constructed, we can prove that $\vdash$ has several crucial structural properties:

Proposition 3 (Structural Properties of $\vdash$).

Let $\vdash$ be classical semantic consequence. $\vdash$ satisfies the following properties:

1. ***Reflexivity**: If $\phi \in \Gamma$, then $\Gamma \vdash \phi$.*
2. ***Monotonicity**: If $\Gamma \vdash \phi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \phi$.*
3. ***Transitivity**: If $\Gamma \vdash \psi$ for all $\psi \in \Delta$, and $\Delta \vdash \phi$, then $\Gamma \vdash \phi$.*
4. ***Disjunction of Premises (Proof by Cases)**: If $\Gamma \cup \{\phi\} \vdash \chi$ and $\Gamma \cup \{\psi\} \vdash \chi$, then $\Gamma \cup \{\phi \vee \psi\} \vdash \chi$.*

We can show that Compactness holds for classical logic. We can give two proofs for it. To prove Definition 1 (Compactness) using the first method, we need the following concepts:

Definition 4 (Well-Behaved Set of Formulas)**.**

Consider $F \subseteq \mathcal{L}$, where $\mathcal{L}$ is the set of well-formed formulas consisting only of $\wedge, \vee, \neg$. The set F is *well-behaved* iff, for all formulas $a, b \in \mathcal{L}$:

1. $a \wedge b \in F \iff a \in F \text{ and } b \in F$,
2. $a \vee b \in F \iff a \in F \text{ or } b \in F$,
3. $\neg a \in F \iff a \notin F$.

Lemma 5.

1. *For any valuation function $v : \mathcal{L} \to \{0, 1\}$, the set $\{a \in \mathcal{L} : v(a) = 1\}$ is well-behaved.*
2. *For all $F \subseteq \mathcal{L}$, if F is well-behaved, then there exists a valuation function v such that $F = \{a \in \mathcal{L} : v(a) = 1\}$.*

Proof.

(1) Consider an arbitrary valuation function $v : \mathcal{L} \to \{1, 0\}$ and the set $V = \{a \in \mathcal{L} : v(a) = 1\}$. Consider arbitrary formulas $a, b \in \mathcal{L}$.

1. $\neg a \in V \iff v(\neg a) = 1 \iff v(a) = 0 \iff a \notin V$.
2. $a \wedge b \in V \iff v(a \wedge b) = 1 \iff v(a) = 1 \text{ and } v(b) = 1 \iff a \in V \text{ and } b \in V$.
3. $a \vee b \in V \iff v(a \vee b) = 1 \iff v(a) = 1 \text{ or } v(b) = 1 \iff a \in V \text{ or } b \in V$.

So, $V = \{a \in \mathcal{L} : v(a) = 1\}$ is well-behaved.

(2) Consider an arbitrary well-behaved $F \subseteq \mathcal{L}$. Let us define $v_F : \mathcal{L} \to \{1, 0\}$ as

$$v_F(a) = \begin{cases} 1 & a \in F \\ 0 & a \notin F \end{cases}$$

and let us show it is a valuation function.

1. First, v_F is a *function*. Take any $a \in \mathcal{L}$. Either $a \in F$ or $a \notin F$. If $a \in F$, $v_F(a) = 1$, and if $a \notin F$, $v_F(a) = 0$. Either way, v_F assigns a unique value (0 or 1) to a, for any $a \in \mathcal{L}$. So v_F is a function.
2. Second, let us show that v_F is an *assignment*. This immediately follows from the step above, for the set of propositional variables $\Phi \subseteq \mathcal{L}$ and v_F is already defined on the entirety of $\mathcal{L}$.

3. Third, let us show that v_F is a *valuation*. It means that the values for complex formulas are assigned according to the usual rules. Consider arbitrary $a, b \in \mathcal{L}$:
    1. $v_F(\neg a) = 1 \iff \neg a \in F \iff a \notin F \iff v_F(a) = 0$.
    2. $v_F(a \wedge b) = 1 \iff a \wedge b \in F \iff a \in F \text{ and } b \in F \iff v_F(a) = 1 \text{ and } v_F(b) = 1$.
    3. $v_F(a \vee b) = 1 \iff a \vee b \in F \iff a \in F \text{ or } b \in F \iff v_F(a) = 1 \text{ or } v_F(b) = 1$.

Therefore v_F is indeed a valuation function. □

Theorem 6 (Compactness for Classical Logic).

The classical consequence relation $\vdash$ (or equivalently the operation Cn) is compact.

Proof. Let us prove the theorem contrapositively. Let $\Gamma \subseteq \mathcal{L}$ and suppose that, for all finite $\Gamma_{fin} \subseteq \Gamma$, $\Gamma_{fin} \nvdash x$. Let us prove that $\Gamma \nvdash x$. We will construct a chain of sets Γ_i resulting in a set Γ^+ such that $\Gamma \subseteq \Gamma^+$, and show that this is the maximal subset of $\mathcal{L}$ such that no finite subset implies x. Then, we will prove that this set is well-behaved, which implies that $\neg x \in \Gamma^+$, and by Lemma 5 that there exists a valuation v such that $v(\Gamma^+) = 1$. We will conclude that since $v(\Gamma^+) = 1$ and $v(x) = 0$, $\Gamma \nvdash x$.

First, note that the set $\mathcal{L}$ is countable, for there is an injective function $\mathrm{G} : \mathcal{L} \to \mathbb{N}$ such that $G(a)$ is the Gödel number of a. So, we can list all the members of $\mathcal{L}$ as $a_1, a_2, a_3, \ldots$. Second, define the following property:

$$P(\Delta) \quad :\iff \quad \text{For all finite } S \subseteq \Delta\text{: } S \nvdash x$$

Let us define the following chain of sets:

$$\begin{array}{ll} 1. & \Gamma_0 := \Gamma \\ 2. & \Gamma_{n+1} := \begin{cases} \Gamma_n \cup \{a_n\} & \text{if } P(\Gamma_n \cup \{a_n\}) \\ \Gamma_n & \text{otherwise} \end{cases} \end{array}$$

Then, define

$$\Gamma^+ := \bigcup_{i=0}^{\infty} \Gamma_i$$

Let us now establish two key properties of Γ^+.

1. **($\Gamma \subseteq \Gamma^+$)** First, $\Gamma \subseteq \Gamma^+$ obviously holds, for $\Gamma = \Gamma_0 \subseteq \Gamma^+$.
2. **($P(\Gamma^+)$ holds)** Second, $P(\Gamma^+)$ holds. Let us first prove by induction that $P(\Gamma_i)$ holds for all $i \geq 0$. First, we have $P(\Gamma_0)$ *ex hypothesi*. Suppose now that $P(\Gamma_n)$ holds (IH), and let us show that $P(\Gamma_{n+1})$ holds. Consider the set $\Gamma_n \cup \{a_n\}$. We have two cases. If $P(\Gamma_n \cup \{a_n\})$ does *not* hold, then $\Gamma_{n+1} = \Gamma_n$, and by the IH $P(\Gamma_{n+1})$ holds. If $P(\Gamma_n \cup \{a_n\})$ holds, $\Gamma_{n+1} = \Gamma_n \cup \{a_n\}$, from which it immediately follows that

$P(\Gamma_{n+1})$ holds. Therefore, $P(\Gamma_i)$ holds for all $i \geq 0$. Suppose now for *reductio* that $P(\Gamma^+)$ does not hold. It follows that there exists a finite $S \subseteq \Gamma^+$ such that $S \vdash x$. Since Γ^+ is the union of a chain of sets $\Gamma \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \ldots$ and S is finite, there must be some Γ_j such that $S \subseteq \Gamma_j$. Since S is finite, $S \vdash x$, and $S \subseteq \Gamma_j$, we have that $P(\Gamma_j)$ does *not* hold. However, this contradicts the fact that $P(\Gamma_i)$ holds for all $i \geq 0$. Therefore, $P(\Gamma^+)$ *must* hold.

3. **(Γ^+ is a maximal P-set)** Let us prove that Γ^+ is the maximal subset of $\mathcal{L}$ satisfying P. Suppose that there exists a set Γ^{++} such that $\Gamma^+ \subset \Gamma^{++}$. We will show that $P(\Gamma^{++})$ fails. Since the inclusion is strict, there exists some $y \in \Gamma^{++}$ such that $y \notin \Gamma^+$. In our enumeration, $y = a_n$ for some n. Since $a_n \notin \Gamma^+$, it implies $a_n \notin \Gamma_{n+1} \subseteq \Gamma^+$. By the definition of our chain, a_n is excluded only if $P(\Gamma_n \cup \{a_n\})$ is false. This means there is a finite set $S \subseteq \Gamma_n \cup \{y\}$ such that $S \vdash x$. Now, observe that $\Gamma_n \subseteq \Gamma^+ \subset \Gamma^{++}$ and $y \in \Gamma^{++}$. So $\Gamma_n \cup \{y\} \subseteq \Gamma^{++}$, and hence, $S \subseteq \Gamma^{++}$. Since Γ^{++} contains a finite subset S that entails x, $P(\Gamma^{++})$ fails. Thus, no proper superset of Γ^+ satisfies P.

Let us now show that Γ^+ is well-behaved.

1. **(Negation: $\neg a \in \Gamma^+ \iff a \notin \Gamma^+$)** ($\Longrightarrow$) Suppose $\neg a \in \Gamma^+$. If $a \in \Gamma^+$, it follows that $\{a, \neg a\} \subseteq \Gamma^+$, and since (*i*) $\{a, \neg a\} \vdash x$ (a contradiction semantically entails everything) and (*ii*) $\{a, \neg a\}$ is finite, $P(\Gamma^+)$ does not hold, contradicting (2) above. Hence $a \notin \Gamma^+$.
   ($\Longleftarrow$) Suppose $a \notin \Gamma^+$. Since Γ^+ is the maximal P-set, $P(\Gamma^+ \cup \{a\})$ does not hold, meaning that there exists some finite $S_1 \subseteq \Gamma^+ \cup \{a\}$ such that $S_1 \vdash x$. Suppose for *reductio* that $\neg a \notin \Gamma^+$. By the maximality of Γ^+, we have that $P(\Gamma^+ \cup \{\neg a\})$ does not hold, meaning that there exists some finite $S_2 \subseteq \Gamma^+ \cup \{\neg a\}$ such that $S_2 \vdash x$. Let $S := (S_1 \setminus \{a\}) \cup (S_2 \setminus \{\neg a\})$. Note, $S \subseteq \Gamma^+$ and S is finite. Note that $S \cup \{a\} = S_1 \cup (S_2 \setminus \{\neg a\})$, and since $S_1 \vdash x$ and $\vdash$ is monotonic, $S \cup \{a\} \vdash x$. Analogously, $S \cup \{\neg a\} = (S_1 \setminus \{a\}) \cup S_2$, and since $S_2 \vdash x$, $S \cup \{\neg a\} \vdash x$. By the Property of Disjunction of Premises – see Proposition 3 (Structural Properties of $\vdash$) – we get $S \cup \{a \vee \neg a\} \vdash x$. Since $a \vee \neg a$ is a tautology, any valuation that satisfies S will trivially satisfy $S \cup \{a \vee \neg a\}$, and thus will satisfy x. Therefore, $S \vdash x$, contradicting $P(\Gamma^+)$. Therefore, $\neg a \in \Gamma^+$.
2. **(Conjunction: $a \wedge b \in \Gamma^+ \iff a, b \in \Gamma^+$)** ($\Longrightarrow$) Suppose $a \wedge b \in \Gamma^+$. If $a \notin \Gamma^+$, by the step above $\neg a \in \Gamma^+$, and hence $\{a \wedge b, \neg a\} \subseteq \Gamma^+$. Since $\{a \wedge b, \neg a\} \vdash x$ and is finite, $P(\Gamma^+)$ does not hold: contradiction. The argument for b is analogous. So, both a, b must be in Γ^+.
   ($\Longleftarrow$) Suppose $a, b \in \Gamma^+$. If $a \wedge b \notin \Gamma^+$, then $\neg(a \wedge b) \in \Gamma^+$. Since $\{a, b, \neg(a \wedge b)\} \vdash x$ and is a finite subset of Γ^+, $P(\Gamma^+)$ does not hold: contradiction. So, $a \wedge b \in \Gamma^+$.
3. **(Disjunction: $a \vee b \in \Gamma^+ \iff a \in \Gamma^+ \text{ or } b \in \Gamma^+$)** ($\Longrightarrow$) Suppose $a \vee b \in \Gamma^+$. If $a \notin \Gamma^+$ and $b \notin \Gamma^+$, it follows that $\neg a, \neg b \in \Gamma^+$. Since $\{a \vee b, \neg a, \neg b\} \vdash x$ and is a finite subset of Γ^+, $P(\Gamma^+)$ does not hold: contradiction.
   ($\Longleftarrow$) Suppose that either $a \in \Gamma^+$ or $b \in \Gamma^+$. Take the former case. Suppose

$a \vee b \notin \Gamma^+$. Hence $\neg(a \vee b) \in \Gamma^+$. Since $\{a, \neg(a \vee b)\} \vdash x$ and is a finite subset of Γ^+, $P(\Gamma^+)$ does not hold: contradiction.

Therefore, Γ^+ is well-behaved in the sense of Definition 4 (Well-Behaved Set of Formulas). By Lemma 5, it follows that there exists a valuation v_{Γ^+} such that $\Gamma^+ = \{a \in \mathcal{L} : v_{\Gamma^+}(a) = 1\}$. This means that $v_{\Gamma^+}(\Gamma^+) = 1$. Moreover, note that $x \notin \Gamma^+$, for if $x \in \Gamma^+$, we have $\{x\} \vdash x$ and clearly $\{x\}$ is a finite subset of Γ^+, violating $P(\Gamma^+)$. Since Γ^+ is well-behaved, $\neg x \in \Gamma^+$, and hence $v_{\Gamma^+}(\neg x) = 1$, meaning $v_{\Gamma^+}(x) = 0$. A fortiori, $v_{\Gamma^+}(\Gamma) = 1$ and $v_{\Gamma^+}(x) = 0$, for $\Gamma \subseteq \Gamma^+$. So, there exists a valuation function v such that $v(\Gamma) = 1$ and $v(x) = 0$, i.e. $\Gamma \nvdash x$. □